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Drift of Buoyant Wing-Tip Vortices

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An exact solution is derived for the trochoidal motion of a single horizontal buoyant vortex drifting under gravity, and a criterion is given for such vortices to persist. Approximate solutions are obtained for the two-dimensional drift of buoyant wing-tip vortices descending in a neutrally stable atmosphere where the buoyancy is achieved by injecting hot gas from the engine exhaust or auxiliary burners into the vortex cores. The vortices are found to approach each other as they descend. This effect may accelerate their breakup and thus reduce the period of time when the wake from a large aircraft is dangerous to other aircraft. At low altitudes the ground effect may dominate and cause buoyant wing-tip vortices to separate instead of approach each other. Formulas and graphs are presented for calculating these effects, and sample calculations are applied to the Boeing 747 aircraft. A weakly stable atmosphere should increase the buoyancy and the convergence rate of descending wing-tip vortices, and a method is presented for calculating the combined effect of core heating and atmospheric stability.

Nomenclature

A = cross-sectional area of a buoyant vortex core which contains the vorticity and buoyant fluid, m^2
a = nominal radius of oval core of a buoyant vortex, such that $A = \pi a^2$, m
b = initial separation of trailing vortices upon roll-up several wing-spans downstream of aircraft, m
g = gravitational acceleration, m/sec^2
P = downward momentum (or impulse) per unit length of a trailing vortex pair, kg-m/sec
p = pressure, N/m^2
$\mathbf{R}$ = (X, Y) position vector in two dimensions of the center of mass of a buoyant vortex core, m
$\mathbf{R}_\omega$ = center of circulation of a vortex, m
$\mathbf{r}$ = (x,y) position vector in two dimensions, m
S = area of integration, m^2
T_F = stagnation temperature of the fan exhaust from a fanjet engine, °K
T_o = temperature of the standard atmosphere, °K
T_P = stagnation temperature of the primary engine exhaust from a fanjet engine, °K
t = time, sec
$\mathbf{U}$ = velocity of displacement of vorticity such that the circulation on any circuit which moves with velocity $\mathbf{U}$ remains constant, m/sec
$\mathbf{v}$ = fluid velocity, m/sec
$\mathbf{v}_\infty$ = uniform flow velocity, m/sec
W_P = mass flux through the primary engine of a fanjet, kg/sec
W_F = mass flux through the fan of a fanjet engine, kg/sec
w = relative forward airspeed, m/sec
X, Y = Cartesian coordinates for the center of mass of a buoyant vortex in two dimensions, m
Y_m = minimum altitude of an aircraft below which the ground effect dominates and buoyancy will not bring its trailing vortices closer together, m
x,y = Cartesian coordinates in two dimensions, m
$\hat{z}$ = unit vector perpendicular to the x-y plane and parallel to the vorticity vector in two dimensions
Γ = circulation of a vortex, m^2/sec
κ = $\Gamma^2/2\pi^2 a^2 bg$ dimensionless parameter
λ = buoyancy index defined by equation (4)
ρ = fluid density, kg/m^3
ρ' = a constant, or a single-valued function of the pressure, kg/m^3
ρ_o = density of the standard atmosphere, kg/m^3
τ = contact time for wing-tip vortices made buoyant by initial core heating in a neutrally stable atmosphere, defined as the time after generation for the vortex cores to drift into uniform tangential contact, sec

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τ' = contact time for wing-tip vortices made buoyant by descent through a weakly stable atmosphere, sec
Φ = $-gy$ gravitational potential, m^2/sec^2
φ = arbitrary scalar function, m^2/sec^2
φ' = scalar function which is constant outside the core of a buoyant vortex, but arbitrary inside, m^2/sec^2
ψ = stream function, m^2/sec
ω = vorticity, sec^{-1}

Introduction

A SUGGESTION was made by the author in Ref. 1, page 191, that if the wing-tip vortices of an aircraft were made buoyant by injecting a lower density fluid into the cores they would tend to approach each other as they descend and possibly disintegrate more quickly. The theoretical basis for this suggestion is presented herein, together with some preliminary calculations. The investigation is applicable to the general problem of controlling aircraft trailing vortices for greater flight safety and increased air traffic density. The trailing vortices are assumed to be made buoyant by heating the cores with exhaust from the aircraft's propulsion engines or from auxiliary kerosene burners on the wing tips. Some of the exhaust from the outboard engines of existing aircraft does enter the trailing vortex cores; but ideally for core heating, the engines would be located on the wing tips, as has also been suggested for drag reduction in reference 2.

Some effects of buoyancy on trailing vortices have been treated by Scorer and Davenport[3] and by Tombach.[4] In these treatments the vortices are initially nonbuoyant and acquire buoyancy during their descent if the atmosphere is stably stratified. Then, the entire oval cross section of the fluid that moves with the vortex pair becomes buoyant, except for external fluid entrained along the upper boundary by the Rayleigh-Taylor instability. The trailing vortices treated herein are initially buoyant because of core heating, and the atmosphere is taken to be neutrally stable. Some consideration is also given to combining the effects of core heating and atmospheric stability. Another departure of the present treatment from that of Refs. 3 and 4 is that the drift of the vortices is derived from an exact equation for the drift of a single buoyant vortex under gravity, instead of being based upon the impulse formula for a classical vortex pair. This exact equation, as derived herein, is a significant advance over the author's earlier presentation in Ref. 5 and should supersede it.

The theory to be presented is not a stability theory like that of Refs. 6–9. It is a theory for the two-dimensional drift of buoyant vortices under gravity—not for their response to small perturbations. A stability theory for buoyant trailing vortices can be developed from the equation of motion to be presented, but this is not done herein.

Equation for the Motion of Distributed Vorticity

Our formulation starts with the equation for vortex motion given on page 12 of Ref. 10, and the vorticity equation given on page 13. When only the terms which are relevant to buoyant vortices are retained, these equations become

$$(\mathbf{U}-\mathbf{v})\times\boldsymbol{\omega}+\nabla p/\rho=\nabla\varphi \tag{1}$$

$$\partial\boldsymbol{\omega}/\partial t-\operatorname{curl}(\mathbf{U}\times\boldsymbol{\omega})=0 \tag{2}$$

In two dimensions (x,y), with $\boldsymbol{\omega}=\hat{z}\omega$, Eq. (2) becomes

$$\partial\omega/\partial t+\operatorname{div}\omega\mathbf{U}=0 \tag{3}$$

and ω and $\mathbf{U}$ are seen to satisfy the same continuity equation in two dimensions as ρ and $\mathbf{v}$. In the form given, Eq. (1) applies to the motion of vorticity in an inviscid nonbarotropic fluid which is subject to conservative forces only (such as gravity).

The density may be written

$$\rho(x,y,t)=\rho'[1-\lambda(x,y,t)] \tag{4}$$

where either $\rho'=$ const or $\rho'=\rho'(p)$ where $\rho'(p)$ is a single-valued function of the pressure p. The dimensionless parameter λ describes the deviation of the fluid from neutral buoyancy and is termed the "buoyancy index." The range of λ is herein restricted to $0\leq\lambda\leq1$. In regions where $\lambda=0$, the fluid is neutrally buoyant, and where $\lambda=1$, the fluid has zero density (maximum buoyancy). It follows from Eq. (4) that

$$1/\rho=1/\rho'+\lambda/\rho \tag{5}$$

and Eq. (1) may be rewritten

$$(\mathbf{U}-\mathbf{v})\times\boldsymbol{\omega}+\lambda\,\nabla p/\rho=\nabla\varphi' \tag{6}$$

where the term $\nabla p/\rho'$ has been absorbed into the arbitrary term, which is now denoted $\nabla\varphi'$.

The momentum equation for an inviscid fluid is

$$D\mathbf{v}/Dt+\nabla p/\rho-\nabla\Phi=0 \tag{7}$$

Solving Eq. (7) for $\nabla p/\rho$ and substituting into Eq. (6), we obtain

$$(\mathbf{U}-\mathbf{v})\times\boldsymbol{\omega}-\lambda\left(\frac{D\mathbf{v}}{Dt}-\nabla\Phi\right)=\nabla\varphi' \tag{8}$$

This equation applies to the motion of distributed vorticity in an inviscid fluid of nonuniform density subject to gravity.

Buoyant Vortex Model

A steady model for an isolated buoyant vortex of circulation Γ in a downward gravitational field is depicted in Fig. 1. The flow is two-dimensional, inviscid, and incompressible, and it is determined from the vorticity ω by the equations

$$\begin{aligned} v_x&=-\partial\psi/\partial y\\ v_y&=\partial\psi/\partial x\\ \partial^2\psi/\partial x^2+\partial^2\psi/\partial y^2&=\omega \end{aligned} \tag{9}$$

The lower density fluid and the vorticity are contained within one of the closed streamlines, as indicated by the dotted region. The vorticity ω within the core is a function of the stream function ψ, so that Kelvin's circulation theorem (Ref. 11, pp. 203–204) is satisfied. Vortex sheets of opposite sense

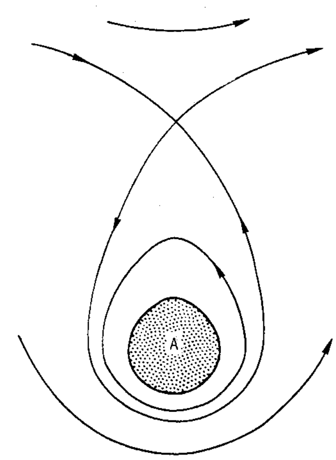

Fig. 1 Conceptual sketch of the steady streamlines for a buoyant vortex in an inviscid, incompressible fluid. The vorticity and lower density fluid are indicated by the dotted region. The fluid within the closed streamline which passes through the stagnation point moves with the vortex.

may occur on opposite sides of the core boundary according to Bjerknes' theorem (Ref. 11, p. 247). Figure 1 is only a conceptual sketch. We assume that some such steady model exists, although its precise form has not yet been determined. Fortunately, the equation of drift for such a vortex can be obtained without knowing the shape of the core or the distribution of vorticity.

Derivation of the Equation for Drift of a Buoyant Vortex

An equation for the drift of an isolated buoyant vortex may be obtained by integrating Eq. (8) over the core cross section. Fig. 1 shows the vortex as viewed from a moving reference frame where the motion is steady. The integration will not be done in this frame—but in the laboratory frame (x,y) of Fig. 2 where the vortex is in motion. Inside the core $\lambda =$ constant; outside the core $\lambda = \omega = 0$. By Eq. (8), therefore, the arbitrary function $\varphi' =$ constant outside the core, although φ' remains arbitrary inside the core. Integration of Eq. (8) over the vortex core of area A yields

$$\iint_A \mathbf{U}\times\boldsymbol{\omega}\,dS - \iint_A \mathbf{v}\times\boldsymbol{\omega}\,dS - \lambda\iint_A \frac{D\mathbf{v}}{Dt}dS + \lambda A\nabla\Phi = 0 \qquad (10)$$

where the integral of $\nabla\varphi'$ vanishes because $\varphi' =$ constant outside the core. If the center of mass (or centroid) of the core fluid $\mathbf{R}$ is defined by

$$\mathbf{R} = \frac{\iint_A \mathbf{r}\rho dS}{\iint_A \rho dS} \qquad (11)$$

the third integral in Eq. (10) may be written

$$\iint_A \frac{D\mathbf{v}}{Dt}dS = A\ddot{\mathbf{R}} \qquad (12)$$

Since the circulation Γ is also conserved according to Eq. (3), we may also define a center of circulation $\mathbf{R}_\omega$ by

$$\mathbf{R}_\omega = \frac{1}{\Gamma}\iint_A \mathbf{r}\omega dS \qquad (13)$$

and the first integral in Eq. (10) may be written

$$\iint_A \mathbf{U}\times\boldsymbol{\omega}\,dS = \dot{\mathbf{R}}_\omega \times \mathbf{\Gamma} \qquad (14)$$

The center of circulation $\mathbf{R}_\omega$ need not coincide with the center of mass $\mathbf{R}$. In general, $\mathbf{R}_\omega$ may revolve and vibrate about $\mathbf{R}$. Neglecting these internal motions, however, we must have $\dot{\mathbf{R}}_\omega = \dot{\mathbf{R}}$ and $\ddot{\mathbf{R}}_\omega = \ddot{\mathbf{R}}$ if the vorticity and the low-density fluid are to remain together during translation of the core. Consequently, Eq. (10) becomes with Eqs. (12) and (14) substituted

$$\dot{\mathbf{R}}\times\mathbf{\Gamma} - \iint_A \mathbf{v}\times\boldsymbol{\omega}\,dS - \lambda A(\ddot{\mathbf{R}} - \nabla\Phi) = 0 \qquad (15)$$

By Eq. (9), the velocity field $\mathbf{v}$ may be written

$$\mathbf{v} = \mathbf{v}_\infty + \mathbf{v}_s \qquad (16)$$

where $\mathbf{v}_\infty$ is a uniform flow and $\mathbf{v}_s$ is the self-velocity field of the vortex. Since $\iint_A \mathbf{v}_s \times \boldsymbol{\omega} dS = 0$, Eq. (15) becomes upon substitution of Eq. (16)

$$(\dot{\mathbf{R}} - \mathbf{v}_\infty)\times\mathbf{\Gamma} - \lambda A(\ddot{\mathbf{R}} - \nabla\Phi) = 0 \qquad (17)$$

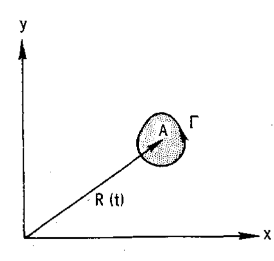

Fig. 2 Buoyant vortex of core area A and circulation Γ drifting with velocity $\dot{\mathbf{R}}$ in laboratory reference frame.

This new result is an exact equation for the two-dimensional drift of a horizontal, inviscid, incompressible buoyant vortex in a uniform flow subject to gravity. Its validity is independent of the shape of the low-density core or of the distribution of vorticity within it. Note that for $\lambda = 0$ (the neutrally buoyant case) we recover Kelvin's result that the vortex moves with the fluid. Equation (17) may be contrasted with Greenhill's equation for the drift of a buoyant circular cylinder with circulation in a fluid subject to gravity, as presented in Ref. 11, pages 79–80. A neutrally buoyant cylinder, of course, need not move with the fluid.

Solution for the Drift of a Single Horizontal Buoyant Vortex

Consider now the motion of a horizontal buoyant vortex in an unbounded fluid which is otherwise at rest and subject to the downward force of gravity. The x and y-components of Eq. (17) for this case where $\mathbf{v}_\infty = 0$ are

$$\dot{Y} - (\lambda A/\Gamma)\ddot{X} = 0 \qquad (18a)$$

$$(\Gamma/\lambda A)\dot{X} + \ddot{Y} + g = 0 \qquad (18b)$$

Solution of these equations shows that the path of the centroid of the vortex core is a trochoid given by

$$X(t) = \alpha - (\lambda A g/\Gamma)t + \gamma\cos[(\Gamma t/\lambda A) + \epsilon] \qquad (19a)$$

$$Y(t) = \beta - \gamma\sin[(\Gamma t/\lambda A) + \epsilon] \qquad (19b)$$

where the constants α, β, γ, and ϵ are determined by the initial conditions. On the average, the vortex drifts to the left with velocity $\lambda A g/\Gamma$.

Persistence of a Horizontal Buoyant Vortex

If the Rayleigh-Taylor instability (Ref. 12, pp. 428–480) should occur on the upper core surface of a horizontal buoyant vortex, buoyant fluid would tend to escape from the core. In order to prevent this, we may enforce the condition that the net outward pressure gradient at all points on the periphery of the core be greater than zero. However, since the exact shape and structure of a buoyant vortex is not known, we shall enforce this condition, instead, on a moving Rankine vortex in the hope that the result thus obtained will approximate that for an actual buoyant vortex. This condition for a moving Rankine vortex of core radius a is

$$(\Gamma/2\pi a^2)\dot{X} + (\Gamma^2/4\pi^2 a^3) - g > 0 \qquad (20)$$

Assuming that this formula also applies approximately to a horizontal buoyant vortex, we shall substitute for $\dot{X}$ the average drift velocity given by Eq. (19a) and thus obtain the condition for persistence

$$\Gamma^2/a^3 g > 4\pi^2 \qquad (21)$$

where a is now the nominal core radius of the buoyant vortex defined by $A = \pi a^2$.

An alternative condition for persistence is that the area of the core A must not exceed the area of the fluid which moves the vortex, as indicated in Fig. 1. Enforcing this condition, again on a moving Rankine vortex, yields another approximate persistence relation for buoyant vortices

$$\Gamma^2/a^3 g > 7.18\pi^2\lambda \qquad (22)$$

For $\lambda < 0.557$, Eq. (21) is the more stringent condition, and vice versa. Hence, these two inequalities may be combined into a single persistence condition for horizontal buoyant vortices

$$\frac{\Gamma^2}{a^3 g} > \begin{cases} 4\pi^2 & (\lambda < 0.557) \\ 7.18\pi^2\lambda & (\lambda > 0.557) \end{cases} \qquad (23)$$

It should be emphasized that condition (23) is based upon a Rankine vortex approximation and on the average drift velocity, with trochoidal oscillations neglected. This condition is intended only to give an order of magnitude for the constraints on the persistence of buoyant vortices. As Scorer and Davenport (Ref. 3, pp. 458–460) have shown, using the Bjerknes equation, vortex sheets of opposite sense may occur on opposite sides of the core boundary. Consequently, the vortex may also be subject to the Kelvin-Helmholtz instability (Ref. 12, pp. 481–514). Although this possibility should be investigated, some of the buoyant vortices which occur naturally (e.g., tilted severe rotating storms) appear to be stable.

Maximum Average Drift Speed for Buoyant Vortices

For buoyant vortices of nominal core radius a and buoyancy index λ, condition (23) may be viewed as stating the minimum allowable circulation for persistence. By Eq. (19a), this corresponds to a maximum average drift speed in a gravitational field given by

$$\langle \dot{X} \rangle_{\max} = \begin{cases} -(\lambda/2)(ag)^{1/2} & (\lambda < 0.557) \\ -(ag\lambda/7.18)^{1/2} & (\lambda > 0.557) \end{cases} \tag{24}$$

Equation (24)—like condition (23)—is an approximate result for a persistent horizontal buoyant vortex.

Drift of Buoyant Trailing Vortices

Equation (17) may be applied to the two-dimensional drift of buoyant trailing vortices depicted in Fig. 3. The component equations for the motion of the centroid of the core of vortex 1 are

$$\dot{Y} + \Gamma/4\pi X - (\lambda A/\Gamma)\ddot{X} = 0 \tag{25a}$$

$$(\Gamma/\lambda A)\dot{X} + \ddot{Y} + g = 0 \tag{25b}$$

where v_∞ has been replaced by the velocity field $-\Gamma/4\pi X$ due to vortex 2. This substitution is valid for $a^2/X^2 \ll 1$, where a is the nominal core radius of the vortices and $2X$ is the center-to-center distance of separation. The substitution breaks down when the vortex cores are near each other for two reasons: a) the velocity field due to vortex 2 then is not approximately constant across the core of vortex 1, and b) the vortex cores become highly elliptical in close proximity, as shown in Ref. 13 with gravity neglected, so that $-\Gamma/4\pi X$ no longer approximates the velocity field of vortex 2. For simplicity, axial flow within the vortex cores is neglected. Although axial flow is known to occur in trailing vortices,[14–16] it should not affect their two-dimensional drift. The coefficients λA and Γ are taken to be constants. This implies that the vortices are assumed to be generated in a neutrally stable atmosphere. Some effects of stable stratification in the atmosphere will be discussed in a subsequent section.

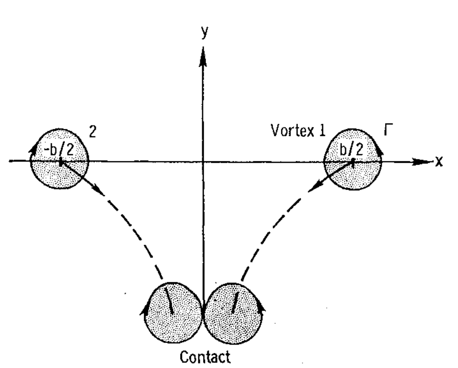

Fig. 3 Drift of buoyant trailing vortex cores. (Rate of approach shown is exaggerated.)

The initial conditions on vortex 1 are that $X(0) = b/2$, $Y(0) = 0$, and $\ddot{X}(0) = \ddot{Y}(0) = 0$. Hence, by Eqs. (25), $\dot{Y}(0) = -\Gamma/4\pi X$ and $\dot{X}(0) = -\lambda A g/\Gamma$. Equation (25b) may be integrated once and, with application of the initial conditions, becomes

$$(\Gamma/\lambda A)(X - b/2) + \dot{Y} + (\Gamma/2\pi b) + gt = 0 \tag{26}$$

Eliminating $\dot{y}$ between Eqs. (26) and (25a) then yields

$$(\Gamma/\lambda A)(X - b/2) + (\Gamma/4\pi)[(2/b) - (1/X)] + gt + (\lambda A/\Gamma)\ddot{X} = 0 \tag{27}$$

If the average drift of the vortices is considered, rather than their short-period trochoidal motion, the $\ddot{X}$ term can be neglected

$$(\Gamma/\lambda A)(X - b/2) + (\Gamma/4\pi)[(2/b) - (1/X)] + gt = 0 \tag{28}$$

Equation (28) may be solved for X as a function of t. However, for our purposes it is expedient to solve for the period of time τ required for the two buoyant vortex cores to drift into uniform contact ($X = a$) after generation, as shown in Fig. 3.

$$\tau = (\Gamma/\pi a g)[(b/2a) - 1][(1/\lambda) + (a/2b)] \tag{29}$$

Equation (29) gives the approximate contact time τ for a pair of buoyant trailing vortices of circulation Γ, buoyancy index λ, nominal radius a, and initial separation after roll-up b.

The contact time τ is actually a hypothetical extrapolation, since the vortex cores are expected to deform and break up before drifting into uniform contact. According to Ref. 17, pages 486–489, a symptom of breakup is that the trailing vortices stop their descent. Therefore, the time when descent ceases may be directly proportional to the contact time τ (or proportional to τ_{combined} as defined by Eq. (40) in the section entitled "Effects of Atmospheric Stability").

Result (29) may also be written in nondimensional form

$$\tau b g/\Gamma = (2/\pi)[(b/2a) - 1][(b/2a\lambda) + \tfrac{1}{4}] \tag{30}$$

The nondimensional contact time $\tau bg/\Gamma$ is plotted against λ for various values of $b/2a$ in Fig. 4. The value $b/2a = 5.076$ is a theoretical result obtained for trailing vortices by Spreiter and Sacks (Ref. 18). But more recent experimental results

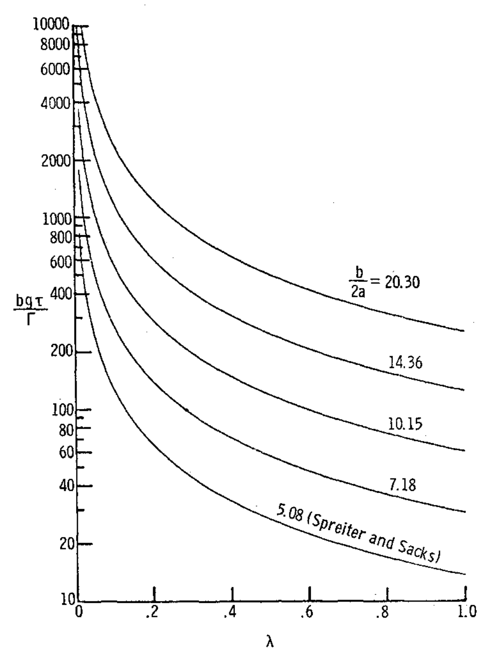

Fig. 4 Plot of the dimensionless contact time $\tau bg/\Gamma$ for buoyant trailing vortices against λ for various values of $b/2a$ [Eq. (30)].

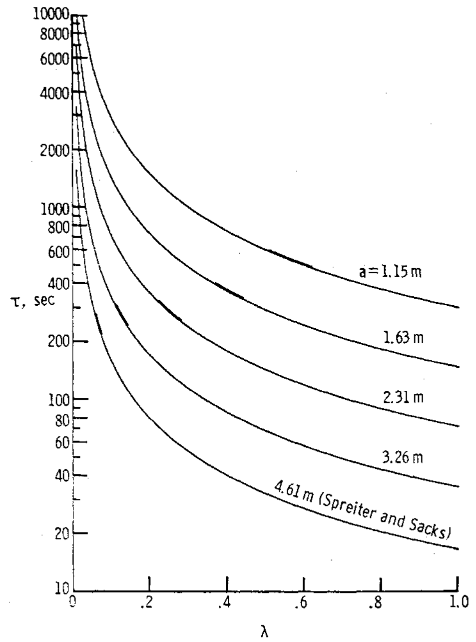

Fig. 5 Plot of the contact time τ for buoyant trailing vortices against λ for various core radii a, with parameters (31) used for the Boeing 747 aircraft. Heavy line segments correspond to buoyancy obtained by heating the vortex cores with the fanjet engine exhaust.

(Refs. 19 and 15) indicate that for a typical wing in clean cruise configuration, the trailing vortex core radius a is only one-fourth or one-fifth as large as the Spreiter and Sacks value. Flaps, spoilers, and wing-tip modifications, however, all enlarge a. Therefore, the curves in Fig. 4 include values of $b/2a$ ranging from the clean wing experimental value to the Spreiter and Sacks value. These curves are valid, of course, only when the persistence relation (23) is satisfied.

Typical contact times are also presented for the Boeing 747 aircraft in Fig. 5, which were obtained by substituting the values

$$\Gamma = 550 \text{ m}^2/\text{sec}$$
$$b = 46.8 \text{ m} \qquad (31)$$
$$g = 9.81 \text{ m/sec}^2$$

If the lifetime is normally about 100 sec (Ref. 17) for the trailing vortices of the Boeing 747 aircraft, then Fig. 5 indicates that buoyancy can decrease this lifetime by a factor of 5.

Example of Wing-Tip Vortices Heated by Fanjet Engine Exhaust

Consider a hypothetical modification to the Boeing 747 aircraft which somehow makes the fanjet exhaust go directly into the cores of the wing-tip vortices. A formula for λ as a function of the engine parameters, core size, and aircraft speed is then given by

$$\lambda = \{W_P[(T_P/T_o) - 1] + W_F[(T_F/T_o) - 1]\}/\pi a^2 w \rho_o \qquad (32)$$

Equation (32) is subject to the constraint that only that portion of W_F and W_P which can be contained within the trailing vortex core of radius a may be used in calculating λ; this constraint takes the form

$$W_P T_P + W_F T_F \leq \pi a^2 \omega \rho_o T_o \qquad (33)$$

Evaluating Eqs. (32) and (33) for flight conditions which are characteristic for the 747 aircraft during takeoff, cruise, and landing (Ref. 20) results in the small range of values plotted along each curve in Fig. 5. With the exception of the curve for $a = 1.154$, where relation (33) limits the core heating, the contact times range from 200 to 400 sec.

One reason that these calculated contact times are not smaller is that the exhaust from the 5:1 bypass ratio fanjet engines used on the 747 aircraft is relatively cool. Substantially lower contact times would be realized for the low bypass ratio, or pure jet, engines proposed for supersonic transports.

Discussion of Trailing Vortex Breakup

As indicated earlier, buoyant trailing vortices are expected to break up before the cores drift into contact. This expectation is plausible because breakup could be caused by the increasing velocity and pressure gradients encountered by each vortex as it approaches the other. Also, according to Ref. 7–9, trailing vortices are subject to a kink instability (the instability of S. C. Crow), which causes the vortex pair to eventually break up into a row of ringlike vortices which then quickly dissipate. It is conceivable that making the vortices buoyant, and thus causing them to approach each other, could cause the Crow instability to occur more quickly.

A slightly different approach is to inject puffs of hot gas at about 6-sec intervals to make the trailing vortices nonuniformly buoyant. Nonuniform buoyancy could cause the vortices to approach each other unevenly and thus trigger the Crow instability and increase its rate of growth. This method appears to have several advantages over the methods presented in references 21 and 22 for accelerating trailing vortex breakup: a) it does not utilize the ailerons or any other control surface of the aircraft, b) it does not affect the flight characteristics of the aircraft, and c) existing aircraft could be modified to use this method by the addition of kerosene burners on the wing tips. Additional fuel would be consumed, of course, but this could be minimized by use only at critical times.

Buoyant Trailing Vortices near the Ground

The image method applied to horizontal trailing vortices near the ground is shown in Fig. 6. The ground effect, which is important near aircraft terminals, tends to make buoyant trailing vortices separate. For a given aircraft there exists a minimum altitude, below which the ground effect is dominant and buoyancy will not bring the trailing vortices closer together. In order to calculate this altitude, we may apply Eq. (17) to vortex 1 of Fig. 6, and substitute for $\mathbf{v}_\infty$ the velocity field due to vortices 2, 3, and 4. The y-component of Eq. (17) then becomes

$$\dot{X} - [\Gamma X^2/4\pi Y(X^2 + Y^2)] + (\lambda A/\Gamma)\ddot{Y} + \lambda Ag/\Gamma = 0 \qquad (34)$$

In order to impose the condition that the vortices do not approach each other after initial roll-up several wingspans

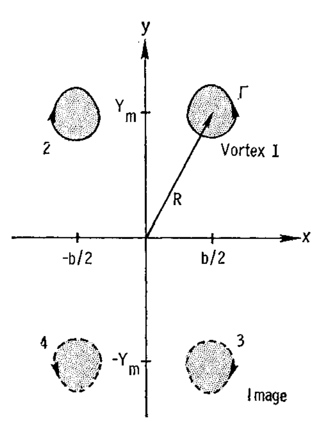

Fig. 6 Buoyant trailing vortices generated at altitude Y_m where their rate of approach is zero.

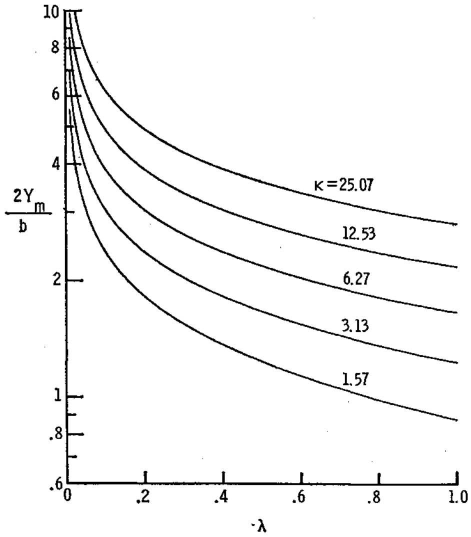

Fig. 7 Plot of dimensionless minimum altitude $2Y_m/b$ for buoyant trailing vortices against λ for various $\kappa \equiv \Gamma^2/2\pi^2 a^2 bg$ [Eq. (36)].

downstream, it is sufficient to stipulate that $\dot{X}(0)=0$ and $Y(0)=Y_m$ along with the other initial conditions that $X(0)=b/2$ and $\ddot{Y}(0)=0$. Substituting these initial conditions into Eq. (34) results in a cubic equation for the minimum altitude Y_m

$$Y_m{}^3 + (b^2/4)Y_m - \Gamma^2 b^2/16\pi^2\lambda a^2 g = 0 \tag{35}$$

In nondimensional form, Eq. (35) may be written

$$(2Y_m/b)^3 + (2Y_m/b) - \Gamma^2/2\pi^2 a^2 bg\lambda = 0 \tag{36}$$

The nondimensional minimum altitude $2Y_m/b$ is plotted against the buoyancy index λ for various values of $\kappa \equiv \Gamma^2/2\pi^2 a^2 bg$ in Fig. 7. For the Boeing 747 aircraft with parameters (31), the minimum altitude Y_m is plotted against λ for various values of the core radius a in Fig. 8. On the same

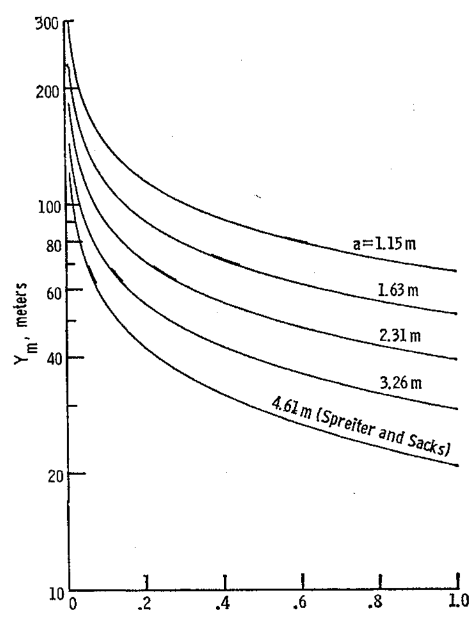

Fig. 8 Plot of minimum altitude Y_m for buoyant trailing vortices against λ for various core radii a with parameters (31) used for the Boeing 747 aircraft. Heavy line segments correspond to buoyancy obtained by heating the vortex cores with the fanjet engine exhaust.

figure are also shown the intervals computed for our hypothetical version of the 747 aircraft, where the fanjet exhaust goes directly into the trailing vortex cores. The range of Y_m thus indicated is from 60 to 80 m.

An effect similar to the ground effect may also apply to buoyant trailing vortices above an inversion layer. Although the ground effect may stop the uniform convergence of buoyant trailing vortices, the alternative suggested earlier of injecting intermittent puffs of hot gas may still promote the Crow instability at altitudes below Y_m. Of course, other methods of vortex control are also available near the ground, such as fences, jets, and suction ditches.

Effects of Atmospheric Stability

The buoyant trailing vortex calculations presented herein apply to a neutrally stable atmosphere. Where the atmosphere is stably stratified, the coefficients λA and Γ in Eq. (17) are no longer constants. Although this important case is not treated herein in detail, several observations can be made about the effects of stable stratification.

Scorer and Davenport[3] and Tombach[4] have considered the two-dimensional drift of trailing vortices which are initially nonbuoyant but become buoyant as they descend through a weakly stable atmosphere. Except for entrainment along the upper boundary, the buoyant fluid is distributed uniformly throughout the nearly elliptical cross section that descends with the vortex pair. Scorer and Davenport conclude that the trailing vortices will approach each other until breakup occurs. However, Tombach, following the treatment by Turner[23] for buoyant smoke plumes, concludes that two patterns are possible: a) if the circulation decays more rapidly than the momentum, the vortices separate as they descend to an equilibrium level, b) if the momentum decays more rapidly than the circulation, the vortices converge as they descend.

From the formulas used in Refs. 4 and 20 for the impulse of a vortex pair P, for the Archimedian force, and for $d\Gamma/dt$ (the Bjerknes equation), it can be shown that

$$(Pd\Gamma/dt)/(\Gamma dP/dt) \leq 0.61 \tag{37}$$

Consequently, it is impossible for the circulation to fall to zero more rapidly than the momentum, and the vortices must converge as they descend in a weakly stable atmosphere, in agreement with the conclusion of Scorer and Davenport.

A contact time may be obtained from the formulas of Scorer and Davenport

$$\tau' = 1.5[(g/\theta)(d\theta/dy)]^{-\frac{1}{2}} \operatorname{sech}^{-1}(2a/b) \tag{38}$$

where θ is the potential temperature. An analogous formula is given by Tombach

$$\tau' = 2.3[-(g/\rho)(d\rho/dy) - g^2/c^2]^{-\frac{1}{2}} \tag{39}$$

where c is the speed of sound and the quantity in the radical is identical to that in Eq. (38). Equations (38) and (39) apply to initially nonbuoyant trailing vortices in a weakly stable atmosphere, while Eq. (29) applies to core-heated trailing vortices in a neutrally stable atmosphere. The two effects should combine harmonically; that is, the contact time for core-heated trailing vortices descending in a weakly stable atmosphere should be given approximately by

$$\tau_{\text{combined}} = \tau\tau'/(\tau + \tau') \tag{40}$$

In these formulas, the weakly stable assumption is important, for, as mentioned earlier, a strong inversion layer could possibly, like the ground, make the vortices separate.

Conclusions

Wing-tip vortices, if made buoyant either by core heating or by descent in a weakly stable atmosphere, should drift towards each other until breakup occurs—except at sufficiently low altitudes where the ground effect dominates and causes the vortices to separate. While this effect could reduce the lifetimes of wing-tip vortices by a factor of 5, core heating from high-bypass ratio fanjet propulsion engines is insufficient to cause this large a reduction.

The two-dimensional drift of a single horizontal buoyant vortex is, in general, trochoidal. However, for the wing-tip vortices considered, the trochoidal oscillation was negligible, its period being less than one second. In order for the lower density fluid to be retained in the core of a horizontal buoyant vortex against the action of gravity, constraints are imposed on the circulation, core size, buoyancy, and drift rate.

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